20 Algebraic representation and formulae

Use letters to represent numbers and express basic arithmetic processes algebraically; substitute numbers for letters; simplify expressions; change the subject of a formula

In algebra we use letters and symbols to represent numbers. We can use algebra to obtain or display a general formula and hence use this formula to solve problems. There are certain rules we must follow in algebra.

First let's look at general algebra notation:

$x + y$	A certain quantity x is added to a different quantity y.
$3x$	3 multiplied by x, e.g. if $x = 2$, $3x = 3 \times (2) = 6$
$4a - 3b$	4 multiplied by a minus 3 multiplied by b
y^2	y to the power of two or $y \times y$, e.g. if $y = 3$, $y^2 = (3)^2 = 9$
$3c^2 + 4d^3$	$3 \times c \times c + 4 \times d \times d \times d$ (3 multiplied by c squared added to 4 multiplied by d cubed).

Whenever possible, algebra should be written in its shortest possible terms – it should be simplified. The process of simplifying algebraic expressions is called **collecting like terms**.

Collecting like terms (simplification)

When adding or subtracting algebraic expressions, you can only **simplify** or combine terms that have the same letter and power of that letter. The final simplified answers should be in alphabetical order with the highest power first (if applicable):

$x + 2x = 3x$ — Letter x has a certain value. If we have one of x added to two xs, we get three xs altogether.

$3y - 2y = y$ — This time we are subtracting two ys from three ys.

$3x + 2y + 2x + 4y = 5x + 6y$ — The x terms can be added to each other and the y terms can be simplified, but we cannot combine them because they are different letters: x represents one quantity and y represents a different quantity.

Collect like terms

$y^2 + 2y^2 + 3y^3 = 3y^3 + 3y^2$ — We can add the first two terms as the letter and power are the same, but the third term has a higher power so it cannot be added to the others.

■ Note that the simplified answers have been arranged in alphabetical order with the highest powers in order from left to right.

Multiplying and dividing algebraic terms

When multiplying terms, first multiply the numbers and then the letters together:

$$3a \times 4b = 3 \times 4 \times a \times b = 12ab$$
$$5c^2 \times 3c^4 = 5 \times 3 \times c \times c \times c \times c \times c \times c = 15c^6$$
$$2c^2 \times 3d^4 \times 2e^2 = 2 \times 3 \times 2 \times c \times c \times d \times d \times d \times d \times e \times e$$
$$= 12c^2d^4e^2$$

When dividing terms, change the terms to their unsimplified form and then cancel where possible:

$$6a^4 \div 2a^2 = \frac{\cancel{6}^3 \times a \times a \times \cancel{a} \times \cancel{a}}{\cancel{2} \times \cancel{a} \times \cancel{a}} = 3 \times a \times a = 3a^2$$

$$10x^5 \div 5x^4 = 2x^1 = 2x$$

$$4x^3y^2 \div x^2y = 4xy$$

A quicker method would be to divide the numbers in your head and use indices rules for the letters, e.g. $x^5 \div x^2 = x^{5-2} = x^3$.

■ Revise the rules of indices in Unit 23 to speed up the process of simplification with powers.

Forming simple algebraic statements

We can choose any letter we like to represent the quantities, but x is used most frequently in algebra.

- the sum of two numbers $x + y$ — x is the first number and y is the second number.
- seven times a number x — $7x$
- half of a number x — $\frac{x}{2}$
- the sum of two numbers x and y divided by a third number z. — $\frac{x+y}{z}$
- the total cost, x, of two apples each costing a pence and three bananas each costing b pence — $x = 2a + 3b$

Substitution in algebra

With substitution, we are told the value of the letters and hence we can calculate the actual value of an expression.

Example 1 If $x = 2,\ y = 3$ and $z = 4$, find the value of the expressions: (a) $3x$ (b) $2y^2$ (c) $2x - z$

(a) $3x = 3 \times (2) = 6$ — $3x$ means three multiplied by the value of x.

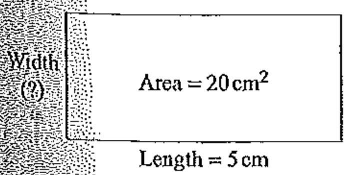
(b) $2y^2 = 2 \times (3)^2 = 2 \times 9 = 18$

(c) $2x - z = 2 \times (2) - (4) = 4 - 4 = 0$

■ Note that the value of the letters can be negative and/or a fraction value. The process of changing the letters to their number value is still the same.

Transforming simple formulae

There are many mathematical formulae in current use, but what if the formula gives one variable and we need another variable somewhere else in the same formula? In these cases we have to rearrange the formula for our required term. Take, for example, the formula used to find the area of a rectangle:



$$\text{Area} = \text{Length} \times \text{Width}$$

Area ↑ Subject of formula — Width ↑ Required term

The formula is used to calculate area when we know the values of length and width. We say that area is the **subject** of this formula. For the rectangle above we know the area and the length but not the width, so we have to rearrange the formula to make the width the subject.

To change the subject of the formula we leave the required term where it is and step by step move terms away from it so that it is isolated on one side and becomes the new subject of the formula.

This is the same as solving an equation for a required unknown.

Example 2 Manipulate the formula $x = cy - d$ so that y is the subject.

$x = cy - d$	We need to eliminate the $-d$ first.
$x + d = cy - d + d$	Add d to both sides.
$x + d = cy$	There are no ds left on the right-hand side.
$\frac{x+d}{c} = \frac{cy}{c}$	Divide both sides by c.
$\frac{x+d}{c} = y$	y is now the subject.

When we move a term from one side to the other, we are effectively inverting its operation:

$+ \rightarrow -$ $\quad - \rightarrow +$

$\times \rightarrow \div$ $\quad \div \rightarrow \times$

$x^2 \rightarrow \sqrt{x}$ $\quad \sqrt{x} \rightarrow x^2$

Changing sides $\rightarrow$ Changing operation

Now we will solve the rectangle problem on page 63.

Area = Length × Width	We can simplify this further by using the first letter for each term.
$A = LW$ L changes sides $\frac{A}{L} = W$	To isolate the W we need to move the L to the other side, remembering to use the inverse operation of multiply by L, which is divide by L. Now W is the subject of the formula.
So $W = \frac{20}{5} = 4\text{ cm}$	Substitute $A = 20$ and $L = 5$, to give $W = 4$ cm.

The width of the rectangle = 4 cm.

Example 3 For each of the following formulae, rearrange to make the letter in **bold** the subject:

(a) $x = \frac{b}{\mathbf{y}}$ (b) $S = \frac{t\mathbf{a}}{p}$ (c) $E = m\mathbf{v}^2$

(a) $x = \frac{b}{y}$	
$xy = b$	Move y to the other side to remove the fraction.
$y = \frac{b}{x}$	Isolate y by moving the x to the other side, remembering to use the inverse of multiply by x, which is divide by x.
(b) $S = \frac{ta}{p}$	
$Sp = ta$	Eliminate the fraction by moving the p to the other side. The inverse operation is multiply by p.
$\frac{Sp}{t} = a$	Isolate a by moving the multiply by t to the other side so it becomes divide by t.
(c) $E = mv^2$	
$\frac{E}{m} = v^2$	Move the multiply by m to the other side to become divide by m.
$\sqrt{\frac{E}{m}} = v$	Remove the square by moving it to the other side where it becomes a square root.

The whole process becomes much quicker with practice. Make sure that you use the correct inverse when you move a term from one side of the equation to the other.

The method of rearranging formulae is called **transposition** and is extremely powerful, as we will see when we look at solving equations in Unit 24. It is used in many other topics throughout the syllabus.

20 Algebraic representation and formulae

Construct equations; transform more complicated formulae

Constructing equations

In order to construct an equation, follow the three simple rules below:

- Represent the quantity to be found by a **symbol**. (x is usually used.)
- Form the **equation** which fits the given information.
- Make sure that both sides of the equation are in the same **units**.

Example 1 Express algebraically: five times a number x minus three times a number y.

$5x - 3y$

Five times x is $5x$ and three times y is $3y$.

Example 2 A girl is m years old now. How old was she 3 years ago?

$m - 3$

Three years before her present age will be m minus 3 years.

Example 3 I think of a number. If I subtract 9 from it and multiply the answer by 4, the result is 32. What is the number I thought of?

$$\begin{aligned} 4(x - 9) &= 32 \\ 4x - 36 &= 32 \\ 4x &= 32 + 36 \\ x &= 17 \end{aligned}$$

Let the unknown number be x. If I subtract nine from it, it becomes $x - 9$. I then multiply it by four, so $x - 9$ goes inside brackets because the whole expression is multiplied by four. The thirty-two goes on the right-hand side of the equation. Then solve as a linear equation.

Example 4 The sides of a triangle are x cm, $(x - 5)$ cm and $(x + 3)$ cm. If the perimeter is 25 cm, find the lengths of the three sides.

$$\begin{aligned} x + (x - 5) + (x + 3) &= 25\,\text{cm} \\ 3x - 2 &= 25\,\text{cm} \\ x &= 9\,\text{cm} \quad \text{(one side)} \\ (x - 5) &= 4\,\text{cm} \quad \text{(second side)} \\ (x + 3) &= 12\,\text{cm} \quad \text{(third side)} \end{aligned}$$

Perimeter of the triangle is the sum of all three sides, so we simply add the three lengths. When we have the first dimension, using substitution, we can find the remaining two sides.

■ Note that we have gone one step further with examples 3 and 4 and solved for x. Solving equations will be looked at in greater detail in Unit 24.

Transforming more complicated formulae

Although the method of 'change sides, change operation' is primarily used again, the formulae are more complicated in the extended syllabus so we need to transform the formulae in the following sequence:

1. Remove square roots or other roots.
2. Remove fractions.
3. Clear brackets.
4. Collect together the terms containing the required subject.
5. Factorise if necessary.
6. Isolate the required subject.

It is unlikely that a formula will contain all six operations, in which case go to the next applicable operation in the list.

Example 5

$M = 5(x + y)$	Make y the subject.
$M = 5x + 5y$	Expand the brackets.
$M - 5x = 5y$	Move the $5x$ to the other side to become $-5x$
$\dfrac{M - 5x}{5} = y$	Move multiply by 5 to the other side to become divide by 5.
$\dfrac{M}{5} - x = y$	Simplify.

Example 6

$y = \dfrac{7}{4 + x}$	Make x the subject.
$(4 + x)y = 7$ $4y + xy = 7$	Remove fractions: move divide by $(4 + x)$ to the other side to become multiply by $(4 + x)$, then expand the brackets.
$xy = 7 - 4y$	Move $+4y$ to the other side to become $-4y$.
$x = \dfrac{7 - 4y}{y}$	Move multiply by y to the other side to become divide by y.
$x = \dfrac{7}{y} - 4$	Simplify.

Example 7

$y = \dfrac{2 - 5x}{2 + 3x}$	Make x the subject.
$(2 + 3x)y = 2 - 5x$	Remove the fraction by taking $(2 + 3x)$ to the other side, and expand the brackets.
$2y + 3xy = 2 - 5x$ $5x + 3xy = 2 - 2y$	Collect the terms involving x on one side; move other terms to the other side.
$x(5 + 3y) = 2 - 2y$	Factorise for x.
$x = \dfrac{2 - 2y}{5 + 3y}$	Move $(5 + 3y)$ to the other side.

■ Note that the correct sequence must be followed: if an operation doesn't apply, then move to the next in sequence.

21 Algebraic manipulation

CORE

Calculate algebra with directed numbers; expand brackets and factorise

In Unit 3 we learnt that directed numbers are numbers with either a positive or a negative sign. When using these numbers in algebra, it is important that we follow the same rules as we would for normal directed number calculations:

(positive number) × / ÷ (positive number) = (positive answer)
(negative number) × / ÷ (negative number) = (positive answer)
(positive number) × / ÷ (negative number) = (negative answer)
(negative number) × / ÷ (positive answer) = (negative answer)

So $3y \times (-2y) = -6y^2$ and $-3y \times -2y = 6y^2$.

Expanding brackets

When removing brackets, every term inside the bracket must be multiplied by whatever is outside the bracket.

Example 1

Expand the brackets and simplify where possible:

(a) $3(x + 2)$ (b) $2a(3a + 4b - 3c)$ (c) $3(x - 4) + 2(4 - x)$

(a) $3(x + 2) = 3x + 6$ — The expression means three lots of x and three lots of two.

(b) $2a(3a + 4b - 3c) = 6a^2 + 8ab - 6ac$ — $2a$ must be multiplied by every term inside the bracket.

(c) $3(x - 4) + 2(4 - x) = 3x - 12 + 8 - 2x$
$= x - 4$ — Expand the first bracket and then the second. Collect together like terms and simplify.

Example 2

Simplify the following expression:

$2(2a + 2b + 2c) - (a + b + c) - 3(a + b + c)$

$= 4a + 4b + 4c - a - b - c - 3a - 3b - 3c$ — Expand each bracket in turn to give all nine terms.

$= 4a - a - 3a + 4b - b - 3b + 4c - c - 3c$
$= 0$ — Collect together the like terms to form one simplified expression.

Factorisation

Factorisation is the reverse of expanding brackets. We start with a simplified expression and put it back into brackets. When factorising, the largest possible factor (number or letter) is removed from each of the terms and placed outside the bracket.

Example 3

Factorise each of the following expressions:

(a) $6x + 15$ (b) $10a + 15b - 5c$ (c) $8x^2y - 4xy^2$

(a) $6x + 15 = 3\left(\frac{6x}{3} + \frac{15}{3}\right)$

3 is the highest number that will divide into both 6 and 15.

$= 3(2x + 5)$

Highest number factor is 3. There is no common letter, so only 3 goes outside the bracket.

(b) $10a + 15b - 5c = 5\left(\frac{10a}{5} + \frac{15b}{5} - \frac{5c}{5}\right)$

5 is the highest number that will divide into 10, 15 and -5.

$= 5(2a + 3b - c)$

Highest number factor is 5. There is no common letter, so only 5 goes outside the bracket.

(c) $8x^2y - 4xy^2 = 4xy\left(\frac{8x^2y}{4xy} - \frac{4xy^2}{4xy}\right)$

$4xy$ is the highest common factor.

$= 4xy(2x - y)$

Example 4

Factorise $4r^3 - 6r^2 + 8r^2s$.

$= 2r^2\left(\frac{4r^3}{2r^2} - \frac{6r^2}{2r^2} + \frac{8r^2s}{2r^2}\right)$

Highest common factor is $2r^2$.

$= 2r^2(2r - 3 + 4s)$

21 Algebraic manipulation

Manipulate harder algebraic expressions, algebraic fractions; factorise using difference of two squares, quadratic, grouping methods

Expanding products of algebraic expressions

These expressions are more complicated, in particular they use double brackets. When expanding double brackets, all terms in the first bracket must be multiplied by all terms in the second bracket.

Example 1 Expand the following and simplify your answer:

(a) $(x+3)(x+2)$ (b) $(2y+1)(2y-2)$ (c) $(4x+4y)^2$

(a) $$\underset{\text{(Bracket 1)}}{(\underset{\substack{\uparrow\\ \text{1st term}\\ \text{(B1)}}}{x} \quad + \quad \underset{\substack{\uparrow\\ \text{2nd term}\\ \text{(B1)}}}{3})} \quad \underset{\text{(Bracket 2)}}{(\underset{\substack{\uparrow\\ \text{1st term}\\ \text{(B2)}}}{x} \quad + \quad \underset{\substack{\uparrow\\ \text{2nd term}\\ \text{(B2)}}}{2})} \begin{aligned} &= x^2+2x+3x+6 \\ &= x^2+5x+6 \end{aligned}$$

All terms in one bracket must be multiplied by each term in the other bracket:

1st (B1) × 1st (B2) = x^2
1st (B1) × 2nd (B2) = $2x$
2nd (B1) × 1st (B2) = $3x$
2nd (B1) × 2nd (B2) = 6

(b) $$\begin{aligned}(2y+1)\,(2y-2) &= 4y^2-4y+2y-2 \\ &= 4y^2-2y-2\end{aligned}$$

Same rules as above, simplifying the answer where possible.

(c) $$\begin{aligned}(4x+4y)^2 &= (4x+4y)(4x+4y) \\ &= 16x^2+16xy+16xy+16y^2 \\ &= 16x^2+16y^2+32xy\end{aligned}$$

The whole of the bracket is squared, so we have to write out the bracket twice and multiply out each term.

Harder factorisation

There are three basic methods:

- factorisation by grouping — Usually involves 4 terms.
- difference of two squares — Involves 2 terms.
- factorisation of quadratic equations. — Involves 3 terms.

Examples 2–4 demonstrate each method.

Factorisation by grouping

Example 2 Factorise the following expressions:

(a) $ax+bx+ay+by$ (b) $6x+xy+6z+zy$ (c) $2x^2-3x+2xy-3y$

(a) $$\begin{aligned}ax+bx+ay+by &= a(x+y)+b(x+y) \\ &= (a+b)(x+y)\end{aligned}$$

Since $(x+y)$ is a factor of both terms we can place a + b inside the other bracket.

(b) $$\begin{aligned}6x+xy+6z+yz &= 6(x+z)+y(x+z) \\ &= (6+y)(x+z)\end{aligned}$$

Now $(x+z)$ is a factor of both terms, so we can place $6+y$ inside the other bracket.

(c) $$\begin{aligned}2x^2-3x+2xy-3y &= 2x(x+y)-3(x+y) \\ &= (2x-3)(x+y)\end{aligned}$$

$(x+y)$ is a common factor, so this goes in one bracket and $2x-3$ goes in the other.

Difference of two squares

If we expand the expression $(x+y)(x-y)$:

$$(x+y)(x-y) = x^2 - xy + xy - y^2 = x^2 - y^2$$

x^2 and y^2 are both squared terms, so $x^2 - y^2$ is known as the **difference of two squares.**

Example 3

Factorise the following:

(a) $4a^2 - 9b^2$ (b) $144 - y^2$ (c) $16x^4 - 81y^4$

(a) $4a^2 - 9b^2 = (2a)^2 - (3b)^2$
$= (2a+3b)(2a-3b)$

In this case the $2a$ term will represent the 'x' and the $3b$ term will represent the 'y'.

(b) $144 - y^2 = (12)^2 - y^2$
$= (12+y)(12-y)$

Now the 12^2 is the first squared term and the y^2 is the second squared term.

(c) $16x^4 - 81y^4 = (4x^2)^2 - (9y^2)^2$
$= (4x^2 + 9y^2)(4x^2 - 9y^2)$

The $(4x^2)^2$ term represents the first squared term and the $(9y^2)^2$ term represents the second squared term.

■ Note that each of the above answers can be checked by reversing the process, i.e. by expanding the brackets back to get the original stated question. The last equation becomes:

$$(4x^2 + 9y^2)(4x^2 - 9y^2) = 16x^4 - 36x^2y^2 + 36x^2y^2 - 81y^4$$
$$= 16x^4 - 81y^4$$

This is the original problem.

Factorisation of quadratic equations

A quadratic equation is any equation which contains a **squared term**, such as $x^2 + 5x + 6$. Any quadratic equation can be written as a product of two brackets.

Example 4

Factorise the following quadratic equations:

(a) $x^2 + 7x + 12$ (b) $x^2 - 2x - 63$ (c) $3x^2 + 8x + 4$

(a) $x^2 + 7x + 12$

We know that first term in each bracket must be x, since x multiplied by $x = x^2$. In order to find the other terms, follow the method below.

x^2 $\quad +7x \quad +12$

$\quad\quad\uparrow\quad\quad\quad\uparrow$

Sum of the two numbers Product of the two numbers

Products which make 12 are 4×3, -4×-3, 2×6, -2×-6, 1×12, -1×-12.

Sums of two numbers which make 7 are $4+3$, $6+1$, $5+2$.

As 4 and 3 are the only numbers that give the correct sum and the correct product, these must be the missing terms.

$$x^2 + 7x + 12 = (x+4)(x+3)$$

■ Note that the order of the numbers is not important.

(b) x^2 $\quad -2x \quad -63$

$\quad\quad\uparrow\quad\quad\quad\uparrow$

Sum Product

Products which make -63 are 9 and -7, 7 and -9, 3 and -21, -21 and 3, -1 and 63, -63 and 1.

Of these, the sum of two numbers which make -2 are 7 and -9.

As 7 and -9 give both the correct sum and the correct product, these must be the missing terms.

$$x^2 + 2x - 63 = (x+7)(x-9)$$

(c) $3x^2$ $+8x$ $+4$

Product of x and $3x$ — Product of two numbers

This type of question is more difficult because we only know that the product is 4 and that the first term in each bracket is x and $3x$, so we have to list the products of 4 and see which two numbers go in the brackets to give us the middle term of $8x$.

Products which make 4 are 2×2, -2×-2, 1×4, -1×-4.

By trial and error, the only two numbers that will give us $8x$ after expanding the brackets are 2 and 2.

$$3x^2 + 8x + 4 = (3x + 2)(x + 2)$$

■ Note that you should always check your answers at the end by multiplying out brackets to get back to the original problem.

Manipulating algebraic fractions

The rules for algebraic fractions are the same as for normal fractions.

For example, when multiplying or dividing:

$$\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

When adding or subtracting, if the denominators are the same, we just add/subtract the numerators.

$$\frac{a}{5} + \frac{b}{5} = \frac{a+b}{5} \qquad \frac{x}{2} - \frac{y}{2} = \frac{x-y}{2}$$

However, when the denominators are different we have to manipulate one of the fractions so that the denominators become equal:

$$\underbrace{\frac{2}{y}}_{\times 2} + \frac{3}{2y} = \frac{4}{2y} + \frac{3}{2y} = \frac{7}{2y}$$

First fraction must be multiplied by 2 to make the denominators equal.

However, in some cases one denominator is not a multiple of the other, so then we have to find a **common multiple** of all the denominators and make this the **common denominator**:

$$\frac{x}{4} + \frac{5y}{9} = \frac{9x}{36} + \frac{20y}{36} = \frac{9x + 20y}{36}$$

36 is a common multiple of 4 and 9.

Even with more complex fractions, the method of finding a common denominator is still required:

$$\frac{1}{x-2} - \frac{2}{x-3}$$

$$= \frac{1}{x-2} \times \frac{x-3}{x-3} - \frac{2}{x-3} \times \frac{x-2}{x-2}$$

Now a common denominator is the product of $x-2$ and $x-3$. We multiply $\frac{1}{x-2}$ by $\frac{x-3}{x-3}$ (which is equal to 1) and multiply $\frac{-2}{x-3}$ by $\frac{x-2}{x-2}$ (which is also equal to 1). Multiplying by 1 in each case means we have not changed the value of the fractions.

$$= \frac{1(x-3) - 2(x-2)}{(x-2)(x-3)}$$

The two fractions now have a common denominator and can be combined and then simplified.

$$= \frac{x - 3 - 2x + 4}{(x-2)(x-3)}$$

$$= \frac{-x + 1}{(x-2)(x-3)}$$

Example 5 Simplify the following algebraic fraction: $\frac{5m}{4y} - \frac{3m}{6y}$

$$\frac{5m}{4y} - \frac{3m}{6y} = \frac{5m(6y)}{(4y)(6y)} - \frac{3m(4y)}{(4y)(6y)}$$

Express the fractions in terms of a common denominator $(4y)(6y)$.

$$= \frac{30my - 12my}{(4y)(6y)}$$

Multiply out the numerators.

$$= \frac{18my}{24y^2}$$

Simplify the fraction to its lowest terms.

$$= \frac{3m}{4y}$$

Example 6 Simplify $\frac{2r}{3x} + \frac{5r}{4x} - \frac{3r}{2x}$

$$\underset{\times 4}{\frac{2r}{3x}} + \underset{\times 3}{\frac{5r}{4x}} - \underset{\times 6}{\frac{3r}{2x}} = \frac{8r}{12x} + \frac{15r}{12x} - \frac{18r}{12x}$$

$$= \frac{5r}{12x}$$

Common denominator $= 12x$, so we multiply each fraction by the appropriate number and then simplify the numerators.

Simplifying algebraic fractions involving quadratics

When you have an expression such as $\frac{x^2 - 2x}{x^2 - 5x + 6}$ factorise the numerator and denominator and then cancel where possible.

Example 7 Simplify the following algebraic fraction: $\frac{x^2 - 2x}{x^2 - 5x + 6}$

$$\frac{x^2 - 2x}{x^2 - 5x + 6} = \frac{x\cancel{(x-2)}}{(x-3)\cancel{(x-2)}}$$

$$= \frac{x}{x-3}$$

$x^2 - 2x$ factorises to $x(x - 2)$.
$x^2 - 5x + 6$ factorises to $(x - 3)(x - 2)$.
Cancel the common terms.

Example 8 Simplifying the following: $\frac{x^2 + 4x}{x^2 + x - 12}$

$$\frac{x^2 + 4x}{x^2 + x - 12} = \frac{x\cancel{(x+4)}}{\cancel{(x+4)}(x-3)}$$

$$= \frac{x}{x-3}$$

$x^2 + 4x$ factorises to $x(x + 4)$.
$x^2 + x - 12$ factorises to $(x + 4)(x - 3)$.
Cancel the common terms.

24 Solution of equations and inequalities

Solve linear equations; solve simultaneous linear equations

Solving linear equations

The method for solving a linear equation is basically the same as for transforming formulae, but we are now given enough information to completely solve it.

Consider the linear equation $x + 4 = 9$

We can rearrange the formula so that x becomes the subject:

$x = 9 - 4 = 5$ which is the solution for x.

This method of 'change sides, change operation' (see Unit 20) is used when solving linear equations, although the equations do get a little more difficult.

Example 1

Solve the following linear equations:

(a) $3x + 4 = 19$ (b) $2(y - 3) = 6$ (c) $6x + 9 = 3x - 54$

(a) $3x + 4 = 19$
$3x = 19 - 4$
$3x = 15$
$x = 5$

Move the +4 to the other side to become −4.

Take the multiply by 3 to the other side to become divide by 3.

(b) $2(y - 3) = 6$
$2y - 6 = 6$
$2y = 6 + 6$
$y = 6$

Expand the brackets, take the −6 to the other side to become +6, then take multiply by 2 to the other side to become divide by 2.

(c) $6x + 9 = 3x - 54$
$6x - 3x = -54 - 9$
$3x = -63$
$x = -21$

Take the $+3x$ to the other side to become $-3x$ and the +9 to the other side to become −9, then simplify. Move the multiply by 3 to the other side to become divide by 3.

Example 2

Solve the equation for x: $\dfrac{7 + 2x}{3} = \dfrac{9x - 1}{7}$

$\dfrac{21(7 + 2x)}{3} = \dfrac{21(9x - 1)}{7}$

Eliminate fractions by multiplying by the LCM (21).

$7(7 + 2x) = 3(9x - 1)$
$49 + 14x = 27x - 3$
$49 + 3 = 27x - 14x$
$52 = 13x$
$4 = x$

Expand brackets, move the xs to one side and numbers to the other side, remembering to invert the operations.

Solving simultaneous equations

This time we are looking for the value of two unknowns, usually x and y. There are two methods for solving simultaneous equations. Both are used in this example.

Example 3 Find the values of x and y in the following simultaneous equations:

$$2x + y = 14 \quad \text{(i)}$$
$$x + y = 9 \quad \text{(ii)}$$

1st method (elimination)

With this method we have to eliminate either the x or the y term by adding or subtracting the equations.

In the equations above we can eliminate the y terms by subtracting the second equation from the first:

$$\begin{array}{r} 2x + y = 14 \\ -x + y = 9 \\ \hline x + 0 = 5 \end{array}$$

Therefore $x = 5$. Now we substitute the value for x we have just found into either of the equations:

Using the first equation:

$$\begin{aligned} 2x + y &= 14 \\ 2(5) + y &= 14 \\ 10 + y &= 14 \\ y &= 4 \end{aligned}$$

So $x = 5$
$y = 4$

2nd method (substitution)

For this method you need to have a good grasp of algebra manipulation.

We have to rearrange one of the equations to make either x or y the subject and then substitute into the other equation.

Choosing the second equation, make x the subject: $x + y = 9$

$$x = 9 - y$$

Now we substitute this expression for x into the first equation:

$$\begin{aligned} 2(9 - y) + y &= 14 \\ 18 - 2y + y &= 14 \\ 18 - y &= 14 \\ 18 - 14 &= y \\ 4 &= y \end{aligned}$$

Now we have the y-value, we substitute this into the second equation to get the x-value:

$$\begin{aligned} x + y &= 9 \\ x &= 5 \end{aligned}$$

Example 4 Solve the following simultaneous equations: $2x = 18 - 5y$ (i)
$x + 3y = 10$ (ii)

Using the substitution method, we rearrange the second equation for x:

$x = 10 - 3y$ — $3y$ has been moved to the other side to become $-3y$.

$2(10 - 3y) = 18 - 5y$ — Now we substitute this expression for x into the other equation to find the value of y.

$20 - 6y + 5y = 18$

$20 - 18 = y$

$2 = y$ — Once we have the value of y we can substitute it into the second equation to find the x-value.

$x + 3(2) = 10$

$x = 4$ — Now we check our values: $2(4) = 18 - 5(2)$

$8 = 8$ Correct

Solving simultaneous equations which have different coefficients

We will now have to manipulate one or both of the equations so that one pair of coefficients is equal and then use the elimination method as before.

Example 5

Solve for x and y: $4x + y = 14$ (i)
$6x - 3y = 3$ (ii)

$3(4x + y) = 3 \times 14$
$12x + 3y = 42$ (iii)

We can make the coefficients of y equal by multiplying the first equation by 3.

Then using the elimination method:

$$\begin{array}{rl} 12x + 3y = 42 & \text{(iii)} \\ +6x - 3y = 3 & \text{(ii)} \\ \hline 18x = 45 & \\ x = 2.5 & \end{array}$$

We add the manipulated equation (iii) to equation (ii). The ys are now eliminated.

Using $x = 2.5$,

$$\begin{aligned} 6(2.5) - 3y &= 3 \\ 15 - 3y &= 3 \\ y &= 4 \end{aligned}$$

Substituting the value of x into equation (ii) finally gives us the value of y.
Don't forget to check!

Example 6

Solve for x and y: $5x - 3y = -0.5$ (i)
$3x + 2y = 3.5$ (ii)

$2(5x - 3y) = 2(-0.5)$
$10x - 6y = -1$ (iii)

$3(3x + 2y) = 3 \times 3.5$
$9x + 6y = 10.5$ (iv)

Multiply the first equation by 2 and the second equation by 3. This gives equal y coefficients.

$$\begin{array}{r} 10x - 6y = -1 \\ +9x + 6y = 10.5 \\ \hline 19x = 9.5 \\ x = 0.5 \end{array}$$

By adding equations (iii) and (iv), the ys are eliminated and x can be found.

$$\begin{aligned} 5(0.5) - 3y &= -0.5 \\ 2.5 - 3y &= -0.5 \\ y &= 1 \end{aligned}$$

Substituting our known value of x into the first equation finally gives us the value of y.

■ Note that we could have eliminated the x coefficients first by making them both equal to 15 (equation (i) multiplied by 3 and equation (ii) multiplied by 5). We would have obtained the same values for x and y.

24 Solution of equations and inequalities

Solve quadratic equations by factorisation and *either* by use of the formula *or* by completing the square; solve simple linear inequalities

Solution of quadratic equations

There are three methods of solving quadratic equations:

- factorisation
- using the formula
- completing the square

We mentioned in Unit 21 (Extended) that a quadratic equation takes the form $ax^2 + bx + c$ where a, b and c are constants. We also looked at the factorising method, so this will not be examined again in detail. But we need to look at how we actually solve a quadratic equation once we have it in brackets form.

Example 1 Solve the equation $x^2 + 3x - 10 = 0$.

The squared term x^2 tells us that the equation has two possible values. First we factorise:

$x^2 + 3x - 10 = 0 \longrightarrow (x + 5)(x - 2) = 0$ Factorising method.

Therefore, if $(x + 5)(x - 2) = 0$ either $(x + 5) = 0$ or $(x - 2) = 0$

So $x = -5$ or $x = 2$

Now perform a check.

If $x = -5$,	If $x = 2$	
$(-5)^2 + 3(-5) - 10 = 0$	$(2)^2 + 3(2) - 10 = 0$	Since the left-hand side of the equation is
$25 - 15 - 10 = 0$	$4 + 6 - 10 = 0$	equal to the right-hand side we know the
$0 = 0$	$0 = 0$	answers obtained for x are correct.

We can solve quadratic equations either by using the completing the square method or by using the quadratic formula. We look at the quadratic formula method on page 81.

The quadratic formula

For a quadratic of the form $ax^2 + bx + c$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{where } a, b \text{ and } c \text{ are the constant values}$$

Example 2 Solve the equation $x^2 + 5x - 7 = 0$, giving your answers to 2 decimal places.

Comparing this to the standard quadratic equation $ax^2 + bx + c$, we see that $a = 1, b = 5, c = -7$.

Using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we can now substitute the values of a, b and c into it:

$$= \frac{-(5) \pm \sqrt{(5)^2 - 4 \times (1) \times (-7)}}{2 \times 1}$$

$$= \frac{-5 + \sqrt{25 + 28}}{2} \quad \text{or} \quad \frac{-5 - \sqrt{25 + 28}}{2}$$

$$= \frac{2.28}{2} \quad \text{or} \quad \frac{-12.28}{2}$$

$$= 1.14 \quad \text{or} \quad -6.14$$

Again we have two possible values for x. Both need to be checked.

If $x = 1.14$

$(1.14)^2 + 5(1.14) - 7 = 0$

$1.2996 + 5.7 - 7 = 0$

$6.9996 - 7 = \text{approx.}\ 0$

Note that the left-hand side doesn't exactly equal zero due to rounding up errors in our calculator but it is very close.

If we do not have the a or b constant values, we would substitute zero for these terms in the formula.

Solving linear inequalities

We looked at simple inequalities in Unit 5 and will see graphical inequalities in Unit 25, so make sure that you are familiar with the symbols.

When solving an inequality treat it as a normal equation, only considering the symbol at the last stage of your working.

Example 3 Solve the inequalities (a) $x + 3 < 7$ (b) $8 \leq x + 1$.

(a) $x + 3 < 7$

$x < 7 - 3$

$x < 4$

Move +3 to other side to become −3, simplify to obtain the solution *and* other possible solutions for x that still satisfy the inequality.

So, x could be 3, 2, 1, . . .

(b) $8 \leq x + 1$

$8 - 1 \leq x$

$7 \leq x$

Move +1 to other side to become −1, simplify, then solve for all possible values of x.

So, x could be 7, 8, 9, . . .

Example 4 Solve the inequality $9 - 4x \geq 17$.

$9 - 4x \geq 17$

$-4x \geq 17 - 9$

$-4x \geq 8$

$x \leq -2$

The +9 is moved to the other side where it becomes −9, the multiply by −4 goes to the other side where it becomes divide by −4.

Invert the inequality to eliminate the minus sign.

So, x could be −2, −3, −4, . . . Possible solutions for x.

Solving harder inequalities

Take, for example, the inequality $5 < 3x + 2 \leq 17$. Here x has a range of values that satisfy the inequalities $3x + 2 > 5$ and $3x + 2 \leq 17$.

Example 5 Find the range of values for the equation $5 < 3x + 2 \leq 17$.

1st part of inequality		**2nd part of inequality**	
$5 < 3x + 2$ $5 - 2 < 3x$ $1 < x$	Move +2 to the other side to become −2, move multiply by 3 to the other side to become divide by 3, and solve for x.	$3x + 2 \leq 17$ $3x \leq 17 - 2$ $x \leq 5$	Move +2 to the other side to become −2, move multiply by 3 to the other side to become divide by 3, and solve for x.

Now we can see that x is both greater than 1 and equal to or less than 5, shown as an inequality:

$1 < x \leq 5$

So, the x-values are $x = 2, 3, 4, 5$.

The process of solving inequalities is very similar to that of solving linear equations, but remember that when you have a negative x-value, make it positive by moving it to the other side of the equation.

When you have to find a range of values as in Example 5, treat the problem as two separate inequalities, solve both and then use your answers to form one general solution(s).