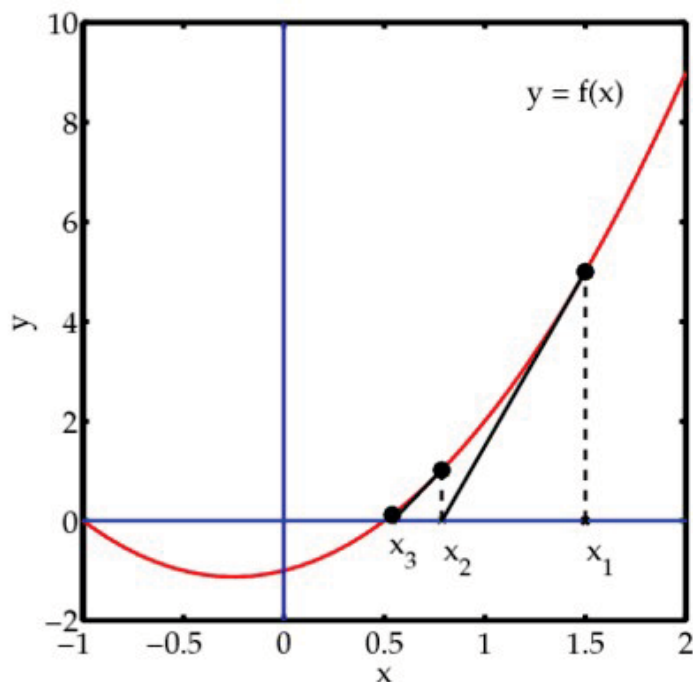


## Maths Exploration

### Newton-Raphson method



**Rationale-** For this project I chose to research and analyse the Newton-Raphson method, where calculus is used to approximate roots. I chose this topic because it looked extremely interesting and the idea of using calculus to approximate roots, seemed intriguing.

The aim of this exploration is to find out how to use the Newton-Raphson method, and in what situations this method is used

#### Explanation of the Newton-Raphson method

The Newton-Raphson or Newton's method is an iterative process to approximate roots. We know simple roots for rational numbers such as  $\sqrt{4}$  or  $\sqrt{9}$ , but what about irrational numbers such as  $\sqrt{3}$  or  $\sqrt{5}$ . This method was discovered in 1736 by Isaac Newton after being published in the 'Method of Fluxions', this method was also described by Joseph Raphson in 1690 in 'Analysis Aequationum'.

#### The Newton-Raphson Process:

In the Newton-Raphson process the following formula is used:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0$$

This is called the Newton-Raphson formula

### How this formula was derived:

The square root of a number ( $n$ ) can be found by using the function:

$$f(x) = x^2 - n$$

The root of  $n$  is the value of  $x$  when  $f(x) = 0$

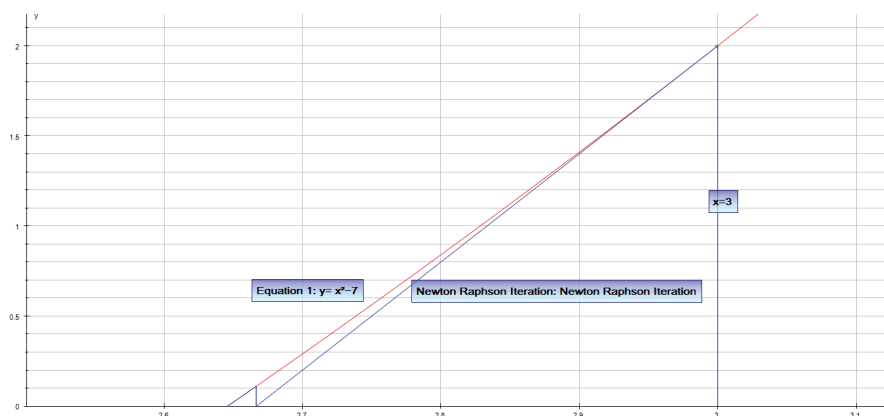
The first thing you do when approximating a root is to make an initial estimate in terms of a positive integer and find the tangent of the function at that point.

Let the initial estimate =  $x_1$

The intercept of the tangent at  $x_1$  with the  $x$ -axis will be closer to the root than the initial estimate. This intercept can be called  $x_2$ .

Calculating  $x_2$  is done by studying the triangle bounded by the  $x$ -axis, the line  $x=x_1$ , and the tangent to the line at  $x_1$ . This only works when the gradient of the function at  $x_1$  is not equal to zero.

An example of a triangle is shown in the following graph (where the square root of 7 is being estimated):



In this case  $x_1=3$

This results in the following formula:

$$x_2 = x_1 - \frac{y_1}{\text{gradient at } x_1}$$

Therefore

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Now that we have a better estimate, the same method can be used again to get closer to the answer. This time  $x_3$  will be found from  $x_2$  represented by the formula below:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$x_3$  will be even closer to the root than  $x_2$  was, and this procedure can be repeated an infinite number of times.

This is an explanation to the Newton-Raphson formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where  $n$  = the number of iterations

The more iterations that are carried out the more accurate the answer is. In practice the iteration is usually only repeated to  $x_{10}$  or less.

For example, if we use this method to approximate the value of  $\sqrt{7}$ , if the equation  $y = x^2 - 7$  is used.

The formula for this function is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

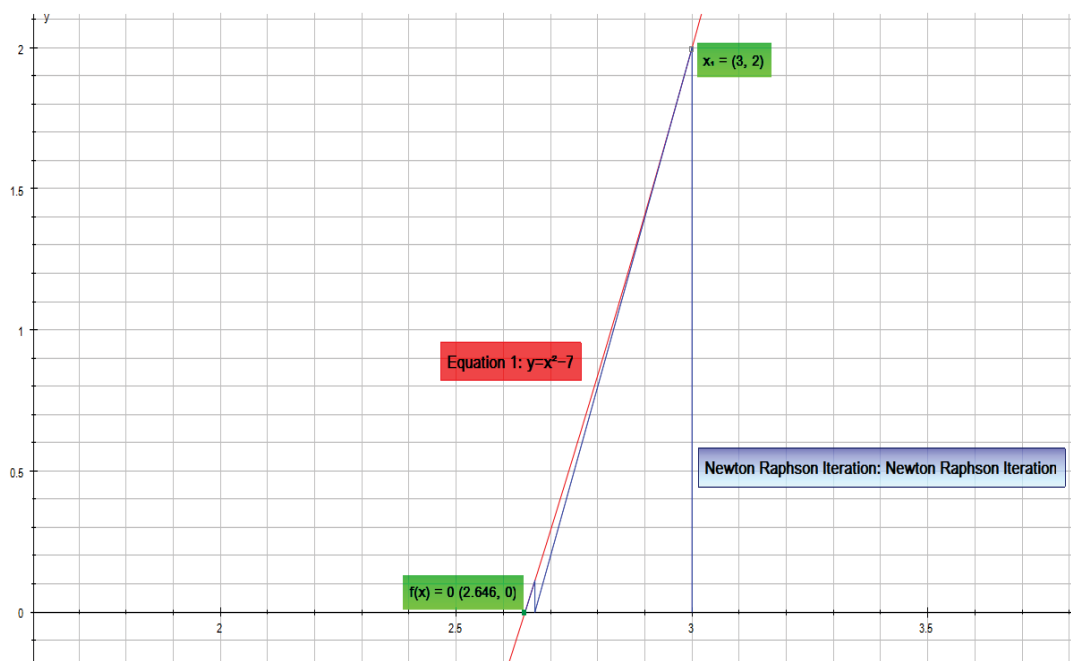
$$x_{n+1} = x_n - \frac{x_n^2 - 7}{2x_n}$$

Using the Newton-Raphson formula for this function the following table showing the value of  $x$  for different numbers of iterations:

$x_1$	3.0000000000
$x_2$	2.6666666667
$x_3$	2.6458333333
$x_4$	2.6457513123
$x_5$	2.6457513111
$x_6$	2.6457513111
$x_7$	2.6457513111
$x_8$	2.6457513111
$x_9$	2.6457513111
$x_{10}$	2.6457513111

The table shows that this method will only need a few iterations and you already have the value of  $x$  to 3 decimal places and after the fifth iteration you have the values of  $x$  up to 10 decimal places.

**Graph showing Newton-Raphson method for  $y = x^2 - 7$ , when approximating  $\sqrt{7}$**



I will now do the same for  $\sqrt{3}$ , and I'll use the equation  $y = x^2 - 3$

The formula for this function is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

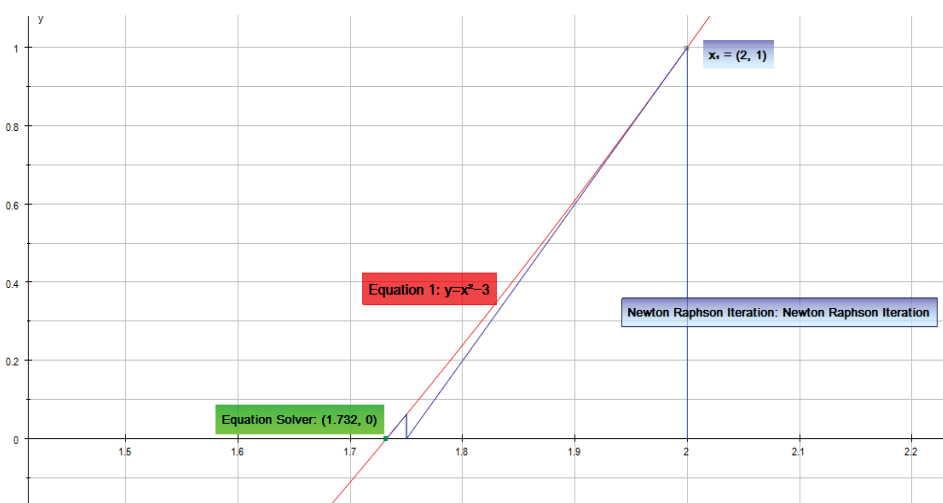
$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}$$

Using the Newton-Raphson formula for this function the following table showing the value of x for different numbers of iterations:

$x_1$	2.0000000000
$x_2$	1.7500000000
$x_3$	1.7321428571
$x_4$	1.7320508100
$x_5$	1.7320508076
$x_6$	1.7320508076
$x_7$	1.7320508076
$x_8$	1.7320508076
$x_9$	1.7320508076
$x_{10}$	1.7320508076

The table again shows that this method will only need a few iterations and you already have the value of x to 3 decimal places and after the fifth iteration you have the values of x up to 10 decimal places.

**Graph showing Newton-Raphson method for  $y = x^2 - 3$ , when approximating  $\sqrt{3}$**



Using a spreadsheet, I repeated the same method for all roots  $\sqrt{x}$  where  $x \in \mathbb{Z}^+$  in the domain  $1 \leq x \leq 7$ , and a table of iterations was produced:

	1	2	3	4	5	6	7
X <sub>1</sub>	1.0000000000	1.5000000000	2.0000000000	2.0000000000	2.5000000000	2.5000000000	3.0000000000
X <sub>2</sub>	1.0000000000	1.4166666667	1.7500000000	2.0000000000	2.2500000000	2.4500000000	2.6666666667
X <sub>3</sub>	1.0000000000	1.4142156863	1.7321428571	2.0000000000	2.2361111111	2.4494897959	2.6458333333
X <sub>4</sub>	1.0000000000	1.4142135624	1.7320508100	2.0000000000	2.2360679779	2.4494897428	2.6457513123
X <sub>5</sub>	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
X <sub>6</sub>	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
X <sub>7</sub>	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
X <sub>8</sub>	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
X <sub>9</sub>	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111
X <sub>10</sub>	1.0000000000	1.4142135624	1.7320508076	2.0000000000	2.2360679775	2.4494897428	2.6457513111

The table shows that after the first few iterations you get a very precise value of  $x$

Although the main function of this method is to approximate roots it can also be used for non-linear equations.

For example the Newton-Raphson method can be used to find when  $\tan(x) = x$  or  $\cos(x) = x$

Where  $f(x) = x - \tan x$  or  $f(x) = x - \cos x$

The Newton Raphson formula can easily be adjusted for these equations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**When**  $f(x) = x - \tan x$

Becomes:

$$x_{n+1} = x_n - \frac{x_n - \tan(x_n)}{1 - \sec^2(x_n)}$$

**And when**  $f(x) = x - \cos x$

Becomes:

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

**Reflection:**

### Evaluation and conclusion

This was a very interesting exploration, as at the start I had no idea what the formula meant and the logic behind it. The idea of using the x-intercept of the gradient to get closer and closer to the correct value of x was amazing. Up to now, I always had a calculator to find the root or zero of a number. If I didn't I would just leave it as the  $\sqrt{3}$  or  $\sqrt{5}$ . Although, this method may not be practical to use on a maths test, it has quite a few advantages. This method gives us the chance to approximate roots up to thousands of decimal places. Even after the first five iterations the value of x is given to 10 decimal places. When finding the root of x, it is quite easy to work out  $x_2$  in your head, which is very close to the precise value. Furthermore, this formula is implemented in technology such as autograph, where Newton-Raphson Iteration is very easy to use and find, as well as by using spreadsheets where it is simple to implement the formula for any number of iterations. Nevertheless, it's easy to find the zero or root by finding the x-intercept, which is much easier than using the Newton-Raphson method on the calculator or any graphing program. This method would be useful though in any field involving zeros or roots where a high-level of precision is needed.

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**Image:**

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