Differentiation Practice Answers

1. (a) $y = \sqrt{3-4x} = (3-4x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3-4x)^{-\frac{1}{2}}(-4)$ (A1)(A1) (C2)

Note: *Award (A1) for each element, to a maximum of [2 marks].*

(b) $y = e^{\sin x}$

$\frac{dy}{dx} = (\cos x)(e^{\sin x})$ (A1)(A1) (C2)

Note: *Award (A1) for each element.*

[4]

2. (a) $\frac{d}{dx}(x^2+1)^2$

$= 2(x^2 + 1) \times (2x)$ (M1)(M1) (C2)

$= 4x(x^2 + 1)$

(b) $\frac{d}{dx}(\ln(3x-1))$

$= \frac{1}{3x-1} \times (3)$ (M1)(M1) (C2)

$= \frac{3}{3x-1}$

[4]

3. $\frac{d}{dx}\left(e^{\frac{x}{3}}\right) = \frac{1}{3}e^{\frac{x}{3}}$ (A1)(A1)

$\frac{d}{dx}(5\cos^2 x) = -10\cos x \sin x$ (A1)(A1)(A1)

$f'(x) = \frac{1}{3}e^{\frac{x}{3}} - 10\cos x \sin x$ (A1) (C6)

[6]

4. **METHOD 1**

$f(x) = 6x^{\frac{2}{3}}$ (A2)

$f'(x) = 4x^{-\frac{1}{3}} \left(= \frac{4}{x^{\frac{1}{3}}} = \frac{4}{\sqrt[3]{x}} \right)$ (A2)(A2) (C6)

METHOD 2

$f(x) = 6(x^2)^{\frac{1}{3}}$ (A1)

$f'(x) = 6 \times \frac{1}{3}(x^2)^{-\frac{2}{3}} \times 2x$ (A2)(A2)

$f'(x) = 4x^{-\frac{1}{3}}$ (A1) (C6)

[6]

5. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \cos 3x$ A1 N1

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\cos^2 x} + \tan x$ accept $x \sec^2 x + \tan x$ A1A1 N2

(c) **METHOD 1**

Evidence of using the quotient rule (M1)

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \times \frac{1}{x} - \ln x}{x^2}$ A1A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - \ln x}{x^2}$ N3

METHOD 2

$y = x^{-1} \text{ In } x$

Evidence of using the product rule (M1)

$\frac{\mathrm{d}y}{\mathrm{d}x} = x^{-1} \times \frac{1}{x} + \ln x(-1)(x^{-2})$ A1A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2} - \frac{\ln x}{x^2}$ N3

[6]

6. (a) $x = \frac{1}{5}$ or $5x - 1 = 0$ (A1) (N1) 1

(b) $f'(x) = \frac{(5x-1)(6x)-(3x^2)(5)}{(5x-1)^2}$ (M1)(A1)

$= \frac{30x^2 - 6x - 15x^2}{(5x-1)^2}$ (may be implied) (A1)

$= \frac{15x^2 - 6x}{(5x-1)^2}$ (accept $a = 15$, $b = -6$) (A1) (N2) 4

[5]

7. (a) $f'(x) = 5e^{5x}$ A1A1 N2

(b) $g'(x) = 2 \cos 2x$ A1A1 N2

(c) $h' = fg' + gf'$ (M1)

$= e^{5x}(2 \cos 2x) + \sin 2x (5e^{5x})$ A1 N2

[6]

8. (a) $f'(x) = 3x^2 - 4x - 0$ (A1)(A1)(A1)

$= 3x^2 - 4x$ (C3)

(b) Gradient $= f'(2)$ (M1)

$= 3 \times 4 - 4 \times 2$ (A1)

$= 4$ (A1) (C3)

[6]

9. (a) $p = 100e^0$ (M1)

$= 100$ (A1) (C2)

(b) Rate of increase is $\frac{dp}{dt}$ (M1)

$\frac{dp}{dt} = 0.05 \times 100e^{0.05t} \quad = 5e^{0.05t}$ (A1)(A1)

When $t = 10$

$\frac{dp}{dt} = 5e^{0.05(10)}$

$= 5e^{0.5} \quad \left(= 8.24 = 5\sqrt{e}\right)$ (A1) (C4)

[6]

10. $f(1) = 1^2 - 3b + c + 2 = 0$ (M1)

$f'(x) = 2x - 3b$,
$f'(3) = 6 - 3b = 0$ (M1)

$3b = 6$, $b = 2$ (A1)

$1 - 3(2) + c + 2 = 0$, $c = 3$ (A1)

Note: *In the event of no working shown, award (C2) for 1 correct answer.*

[4]

11. (a) $\dfrac{f(5+h) - f(5)}{h} = \dfrac{(5.1)^3 - 5^3}{0.1}$

$= 76.51$ (or 76.5 to 3 sf) (A1) (C1)

(b) $\lim_{h \to 0} \dfrac{f(5+h) - f(5)}{h} = f'(5)$ (M1)

$= 3(5)^2$ (A1)

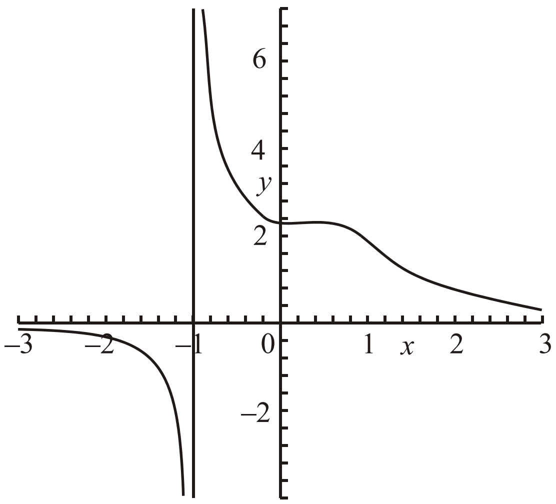
$= 75$ (A1) (C3)

[4]

12. (a) (i) Vertical asymptote $x = -1$ (A1)

(ii) Horizontal asymptote $y = 0$ (A1)

(iii)



(A1)(A1)

Note: *Award (A1) for each branch.*

4

(b) (i) $f'(x) = \dfrac{-6x^2}{\left(1+x^3\right)^2}$

$$f''(x) = \frac{\left(1+x^3\right)^2\left(-12x\right)+6x^2(2)(1+x^3)^1\left(3x^2\right)}{\left(1+x^3\right)^4} \qquad \text{(M1)}$$

$$= \frac{\left(1+x^3\right)\left(-12x\right)+36x^4}{\left(1+x^3\right)^3} \qquad \text{(A1)}$$

$$= \frac{-12-12x^4+36x^4}{\left(1+x^3\right)^3} \qquad \text{(A1)}$$

$$= \frac{12x\left(2x^3-1\right)}{\left(1+x^3\right)^3} \qquad \text{(AG)}$$

(ii) Point of inflexion => $f''(x) = 0$ (M1)

=> $x = 0$ or $x = \sqrt[3]{\dfrac{1}{2}}$

$x = 0$ or $x = 0.794$ (3 sf) (A1)(A1)

OR

$x = 0, x = 0.794$ (G1)(G2) 6

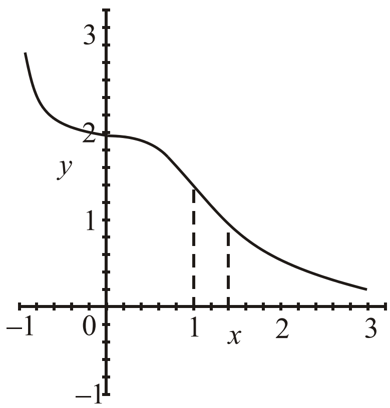
(c) (i) Approximate value of $\int_1^3 f(x)\,\mathrm{d}x$, $h = \dfrac{b-a}{n} = \dfrac{2}{5}$ (A1)

$= \dfrac{1}{5}$ [1 + 1.068377 + ... + 0.215332 + 0.071429] (A1)

$= \dfrac{1}{5}$ (3.284025)

= 0.656805 (A1)

(ii) $\int_1^3 f(x)\,\mathrm{d}x = 0.637599$



(A1)

Between 1 and 3, the graph is 'concave up', so that the straight lines forming the trapezia are all **above** the graph. (R1) 5

[15]

13. (a) $x = 1$ (A1) 1

(b) (i) $f(-1000) = 2.01$ (A1)

(ii) $y = 2$ (A1) 2

(c) $f'(x) = \dfrac{(x-1)^2(4x-13) - 2(x-1)(2x^2 - 13x + 20)}{(x-1)^4}$ (A1)(A1)

$= \dfrac{(4x^2 - 17x + 13) - (4x^2 - 26x + 40)}{(x-1)^3}$ (A1)

$= \dfrac{9x - 27}{(x-1)^3}$ (AG) 3

Notes: *Award (M1) for the* ***correct*** *use of the quotient rule, the first (A1) for the placement of the correct expressions into the quotient rule.*
Award the second (A1) for doing sufficient simplification to make the given answer reasonably obvious.

(d) $f'(3) = 0$ $\Rightarrow$ stationary (or turning) point (R1)

$f''(3) = \dfrac{18}{16} > 0$ $\Rightarrow$ minimum (R1) 2

(e) Point of inflexion $\Rightarrow f''(x) = 0 \Rightarrow x = 4$ (A1)
$x = 4 \Rightarrow y = 0$ $\Rightarrow$ Point of inflexion = (4, 0) (A1)

OR

Point of inflexion = (4, 0) (G2) 2

[10]

14. (a) $y = 0$ (A1) 1

(b) $f'(x) = \dfrac{-2x}{(1+x^2)^2}$ (A1)(A1)(A1) 3

(c) $\dfrac{6x^2 - 2}{(1+x^2)^3} = 0$ (or sketch of $f'(x)$ showing the maximum) (M1)

$6x^2 - 2 = 0$ (A1)

$x = \pm\sqrt{\dfrac{1}{3}}$ (A1)

$x = \dfrac{-1}{\sqrt{3}} (= -0.577)$ (A1) (N4) 4

(d) $\displaystyle\int_{-0.5}^{0.5} \frac{1}{1+x^2}\,dx \left(= 2\int_{0}^{0.5} \frac{1}{1+x^2}\,dx = 2\int_{-0.5}^{0} \frac{1}{1+x^2}\,dx \right)$ (A1)(A1) 2

[10]

15. (a) $f'(x) = 2xe^{-x} - x^2e^{-x}$ $\quad (= (2x - x^2)e^{-x} = x(2 - x)e^{-x})$ A1A1 N2

(b) Maximum occurs at $x = 2$ (A1)

Exact maximum value $= 4e^{-2}$ A1 N2

(c) For inflexion, $f''(x) = 0$ $\quad \left(\left(x^2 - 4x + 2\right) = 0,\ x = \dfrac{4 \pm \sqrt{16-8}}{2}, \text{etc.}\right)$ M1

$x = \dfrac{4+\sqrt{8}}{2}\ \left(= 2 + \sqrt{2}\right)$ A1 N1

[6]

16. (a) $x = 1$ (A1) 1

(b) Using quotient rule (M1)

Substituting correctly $g'(x) = \dfrac{(x-1)^2(1) - (x-2)[2(x-1)]}{(x-1)^4}$ A1

$= \dfrac{(x-1) - (2x-4)}{(x-1)^3}$ (A1)

$= \dfrac{3-x}{(x-1)^3}$ (Accept $a = 3$, $n = 3$) A1 4

(c) Recognizing at point of inflexion $g''(x) = 0$ M1
$x = 4$ A1

Finding corresponding y-value $= \dfrac{2}{9} = 0.222$ *ie* $P\left(4, \dfrac{2}{9}\right)$ A1 3

[8]

17. (a) $y = e^{2x}\cos x$

$\dfrac{dy}{dx} = e^{2x}(-\sin x) + \cos x\,(2e^{2x})$ (A1)(M1)

$= e^{2x}(2\cos x - \sin x)$ (AG) 2

(b) $\dfrac{d^2y}{dx^2} = 2e^{2x}(2\cos x - \sin x) + e^{2x}(-2\sin x - \cos x)$ (A1)(A1)

$= e^{2x}(4\cos x - 2\sin x - 2\sin x - \cos x)$ (A1)

$= e^{2x}(3\cos x - 4\sin x)$ (A1) 4

(c) (i) At P, $\frac{d^2y}{dx^2} = 0$ (R1)

$\Rightarrow 3\cos x = 4\sin x$ (M1)

$\Rightarrow \tan x = \frac{3}{4}$

At P, $x = a$, *ie* $\tan a = \frac{3}{4}$ (A1)

(ii) The gradient at any point $e^{2x}(2\cos x - \sin x)$ (M1)

Therefore, the gradient at P $= e^{2a}(2\cos a - \sin a)$

When $\tan a = \frac{3}{4}$, $\cos a = \frac{4}{5}$, $\sin a = \frac{3}{5}$ (A1)(A1)

(by drawing a right triangle, or by calculator)

Therefore, the gradient at P $= e^{2a}\left(\frac{8}{5} - \frac{3}{5}\right)$ (A1)

$= e^{2a}$ (A1) 8

[14]

18. (a) $x = 1$ (A1)

EITHER

The gradient of $g(x)$ goes from positive to negative (R1)

OR

$g(x)$ goes from increasing to decreasing (R1)

OR

when $x = 1$, $g''(x)$ is negative (R1) 2

(b) $-3 < x < -2$ **and** $1 < x < 3$ (A1)

$g'(x)$ is negative (R1) 2

(c) $x = -\frac{1}{2}$ (A1)

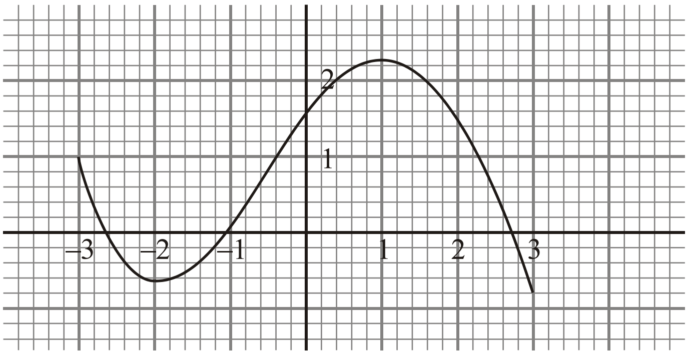
EITHER

$g''(x)$ changes from positive to negative (R1)

OR

concavity changes (R1) 2

(d)



(A3) 3

[9]

19. (a)

	A	B	E
$f'(x)$	negative	0	negative

A1A1A1 N3

(b)

	A	B	E
$f''(x)$	positive	positive	negative

A1A1A1 N3

[6]

20. (a)

Interval	g'	g''
$a < x < b$	**positive**	**positive**
$e < x < f$	**negative**	**negative**

A1A1

A1A1 N4

(b)

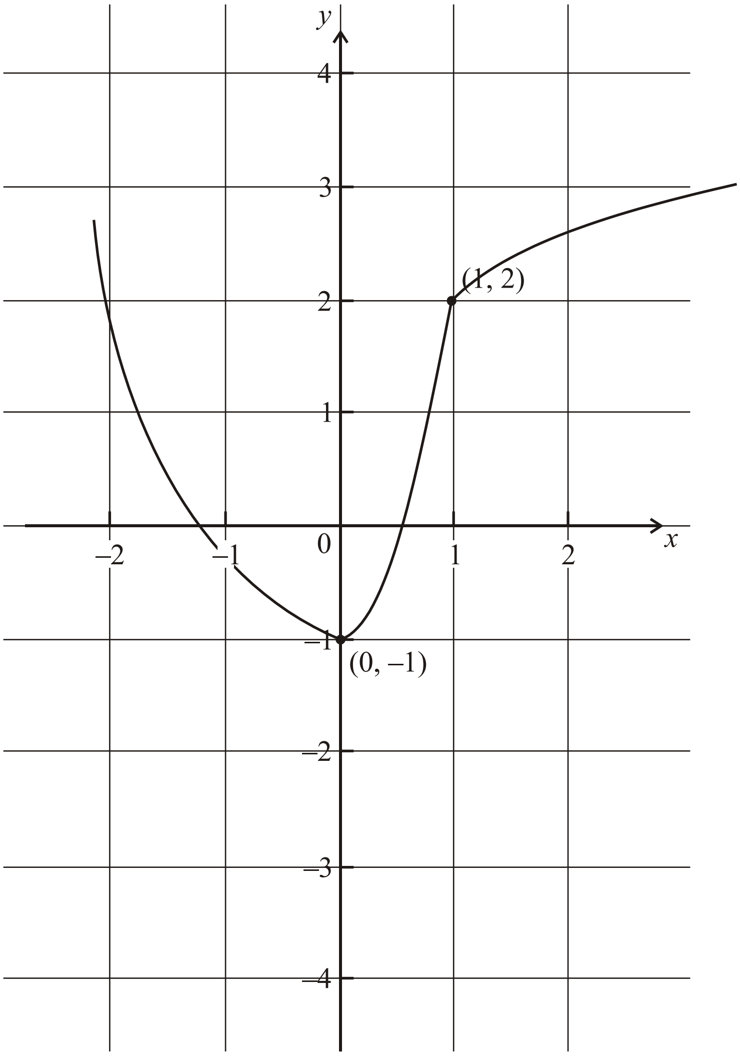
Conditions	Point
$g'(x) = 0, g''(x) < 0$	**C**
$g'(x) < 0, g''(x) = 0$	**D**

A1 N1

A1 N1

[6]

21.



A1A1A1A1A1A1 N6

Notes: *On interval [– 2,0], award A1 for decreasing, A1 for concave up.*

On interval [0,1], award A1 for increasing, A1 for concave up.

On interval [1,2], award A1 for change of concavity, A1 for concave down.

[6]

22. $y = x^3 + 1$

$\frac{dy}{dx} = 3x^2$

= Slope of tangent at any point

Therefore at point where $x = 1$, slope = 3 (M1)

$\Rightarrow$ Slope of normal $= -\frac{1}{3}$ (M1)(A1)

$\Rightarrow$ Equation of normal: $y - 2 = -\frac{1}{3}(x - 1)$

$3y - 6 = -x + 1$

$x + 3y - 7 = 0$ (A1) (C4)

Note: *Accept equivalent forms eg* $y = -\frac{1}{3}x + 2\frac{1}{3}$

[4]

23. (a) $y = x(x - 4)^2$

(i) $y = 0 \Leftrightarrow x = 0$ or $x = 4$ (A1)

(ii) $\frac{dy}{dx} = 1(x - 4)^2 + x \times 2(x - 4) = (x - 4)(x - 4 + 2x)$

$= (x - 4)(3x - 4)$ (A1)

$\frac{dy}{dx} = 0 \Rightarrow x = 4$ or $x = \frac{4}{3}$ (A1)

$$\left.\begin{array}{l} x = 1 \Rightarrow \frac{dy}{dx} = (-3)(-1) = 3 > 0 \\ x = 2 \Rightarrow \frac{dy}{dx} = (-2)(2) = -4 < 0 \end{array}\right\} \Rightarrow \frac{4}{3} \text{ is a } \textbf{maximum}$$ (R1)

Note: *A second derivative test may be used.*

$x = \frac{4}{3} \Rightarrow y = \frac{4}{3} \times \left(\frac{4}{3} - 4\right)^2 = \frac{4}{3} \times \left(\frac{-8}{3}\right)^2 = \frac{4}{3} \times \frac{64}{9} = \frac{256}{27}$

$\left(\frac{4}{3}, \frac{256}{27}\right)$ (A1)

Note: *Proving that* $\left(\frac{4}{3}, \frac{256}{27}\right)$ *is a maximum is not necessary to receive full credit of [4 marks] for this part.*

(iii) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\big((x-4)(3x-4)\big) = \dfrac{d}{dx}(3x^2 - 16x + 16) = 6x - 16$ (A1)

$\dfrac{d^2y}{dx^2} = 0 \Leftrightarrow 6x - 16 = 0$ (M1)

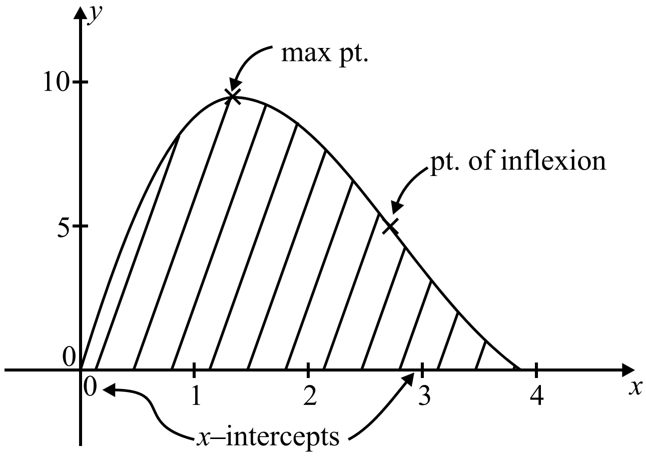
$\Leftrightarrow x = \dfrac{8}{3}$ (A1)

$x = \dfrac{8}{3} \Rightarrow y = \dfrac{8}{3}\left(\dfrac{8}{3} - 4\right)^2 = \dfrac{8}{3}\left(\dfrac{-4}{3}\right)^2 = \dfrac{8}{3} \times \dfrac{16}{9} = \dfrac{128}{27}$

$\left(\dfrac{8}{3}, \dfrac{128}{27}\right)$ (A1) 9

Note: *GDC use is likely to give the answer (1.33, 9.48). If this answer is given with no explanation, award (A2), If the answer is given with the explanation "used GDC" or equivalent, award full credit.*

(b)



(A3) 3

Note: *Award (A1) for intercepts, (A1) for maximum and (A1) for point of inflexion.*

(c) (i) See diagram above (A1)

(ii) $0 < y < 10$ for $0 \le x \le 4$ (R1)

So $\int_0^4 0\,dx < \int_0^4 y\,dx < \int_0^4 10\,dx \Rightarrow 0 < \int_0^4 y\,dx < 40$ (R1) 3

[15]

24. $y = x^2 - x$

$\frac{dy}{dx} = 2x - 1$ = gradient at any point. (M1)

Line parallel to $y = 5x$

$\Rightarrow 2x - 1 = 5$ (M1)

$x = 3$ (A1)

$y = 6$ (A1)

Point (3, 6) (C2)(C2)

[4]

25. (a) π (3.14) (accept $(\pi, 0)$, $(3.14, 0)$) A1 N1

(b) (i) For using the product rule (M1)

$f'(x) = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$ A1A1 N3

(ii) At B, $f'(x) = 0$ A1 N1

(c) $f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x$ A1A1

$= 2e^x \cos x$ AG N0

(d) (i) At A, $f''(x) = 0$ A1 N1

(ii) Evidence of setting up **their** equation (may be seen in part (d)(i)) A1

eg $2e^x \cos x = 0$, $\cos x = 0$

$x = \frac{\pi}{2}\ (=1.57)$, $y = e^{\frac{\pi}{2}}\ (=4.81)$ A1A1

Coordinates are $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ $(1.57, 4.81)$ N2

(e) (i) $\int_0^{\pi} e^x \sin x\, dx$ or $\int_0^{\pi} f(x)\, dx$ A2 N2

(ii) Area = 12.1 A2 N2

[15]

26. $y = \sin(2x - 1)$

$\frac{dy}{dx} = 2\cos(2x - 1)$ (A1)(A1)

At $\left(\frac{1}{2}, 0\right)$, the gradient of the tangent $= 2\cos 0$ (A1)

$= 2$ (A1) (C4)

[4]

27. (a) $f'(x) = 6x - 5$ A1 N1

(b) $f'(p) = 7$ (or $6p - 5 = 7$) M1

$p = 2$ A1 N1

(c) Setting $y(2) = f(2)$ (M1)

Substituting $y(2) = 7 \times 2 - 9$ $(= 5)$, **and** $f(2) = 3 \times 2^2 - 5 \times 2 + k$ $(= k + 2)$ A1

$k + 2 = 5$

$k = 3$ A1 N2

[6]

28. (a) **METHOD 1**

$f'(x) = -6\sin 2x + 2\sin x\cos x$ A1A1A1

$= -6\sin 2x + \sin 2x$ A1

$= -5\sin 2x$ AG N0

METHOD 2

$\sin^2 x = \frac{1 - \cos 2x}{2}$ (A1)

$f(x) = 3\cos 2x + \frac{1}{2} - \frac{1}{2}\cos 2x$ A1

$f(x) = \frac{5}{2}\cos 2x + \frac{1}{2}$ A1

$f'(x) = 2\left(\frac{5}{2}\right)(-\sin 2x)$ A1

$f'(x) = -5\sin 2x$ AG N0

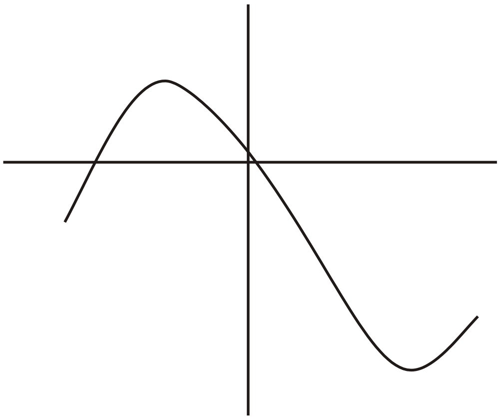
(b) $k = \frac{\pi}{2}$ $(= 1.57)$ A2 N2

[6]

29. (a) **EITHER**

Recognizing that tangents parallel to the x-axis mean maximum and minimum (may be seen on sketch) R1

Sketch of graph of f M1



OR

Evidence of using $f'(x) = 0$ M1

Finding $f'(x) = 3x^2 - 6x - 24$ A1

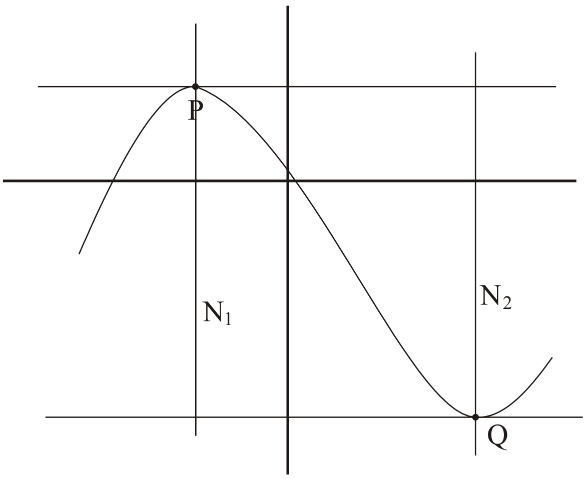
$3x^2 - 6x - 24 = 0$

Solutions $x = -2$ or $x = 4$

THEN

Coordinates are P(−2, 29) and Q(4, −79) A1A1 N1N1

(b)



(i) (4, 29) A1 N1

(ii) (−2, −79) A1 N1

[6]

30. **METHOD 1**

$l + 2w = 60$ (M1)

$l = 60 - 2w$ (A1)

$A = w(60 - 2w) \quad (= 60w - 2w^2)$ (A1)

$\frac{\mathrm{d}A}{\mathrm{d}w} = 60 - 4w$ (A1)

Using $\frac{\mathrm{d}A}{\mathrm{d}w} = 0 \quad (60 - 4w = 0)$ (M1)

$w = 15$ (A1) (C6)

METHOD 2

$w + 2l = 60$ (A1)

$w = 60 - 2l$ (A1)

$A = l(60 - 2l) \quad (= 60l - 2l^2)$ (A1)

$\frac{\mathrm{d}A}{\mathrm{d}l} = 60 - 4l$ (A1)

Using $\frac{\mathrm{d}A}{\mathrm{d}l} = 0 \quad (60 - 4l = 0)$ (M1)

$l = 15$

$w = 30$ (A1) (C6)

[6]